Using multiple attractor chaotic systems for communication

T. L. Carroll and L. M. Pecora
Code 6343
US Naval Research Lab
Washington, DC 20375

In recent work with symmetric chaotic systems, we synchronized two such systems with one-way driving. The drive system had 2 possible attractors, but the response system always synchronized with the drive system. In this work, we show how we may combine 2 attractor chaotic systems with a multiplexing technique first developed by Tsimring and Suschick to make a simple communications system. We note that our response system is never synchronized to our drive system (not even in a generalized sense), but we are still able to transmit information. We characterize the performance of the communications system when noise is added to the transmitted signal.

Introduction

It has been suggested recently that chaotic systems might be useful for communications [1-13]. There are many practical problems that arise when a chaotic signal is transmitted. Among these problems is additive noise. In some communications schemes, a small information signal modulates a parameter in a chaotic system (or is added to the transmitted chaotic signal). The lack of synchronization at the receiver indicates the presence of the information signal, and the synchronization error is used to recover the information. When the information signal is small, it should not take much noise to obscure the information.

In conventional digital communications systems, one tries to decide which of several symbols has been transmitted in a noisy environment using the principal of maximum likelihood [14]. If there are several possible symbols that might have been transmitted, the most likely symbol is taken to be the received symbol. Naturally, this estimation is easier if the symbols are far apart in some symbol space. For our chaotic communications system, we use two widely separated attractors for our two symbols. We then combine signals from two chaotic systems so that our transmitted signal has no DC component. We follow this procedure with two different chaotic systems, and compare signal to noise performance.

Multiple attractor systems

The basic principle that we will use has been described previously [15] in a 4-dimensional circuit. The circuit had a symmetric nonlinearity, so that for some parameters the circuit had two symmetric attractors. We built a drive circuit which drove a response circuit through a one-way driving. Normally, one would expect that the response circuit would also have two attractors, so that the response would not synchronize to the drive unless the response circuit was in the correct basin of attraction. For some parameters, however, the out-of-sync attractor in the response circuit was near neutral stability. After a few cycles in the out-of-sync attractor, the response system converged to the in sync attractor.

The response circuit recognized which attractor the drive circuit was in based only on the transmitted signal. In principle, one could use the two drive circuit attractors for communications symbols. Since the attractors were well separated, it should be easy to distinguish the two attractors in a noisy environment.

One practical problem is that the transmitted signal can contain no DC components. It may be difficult to find a multi-attractor system where the transmitted signal meets this requirement. We can, however, design chaotic systems where the transmitted signal has no DC component. Tsimring and Suschick [12] showed in chaotic maps one could add signals from several chaotic systems and use chaotic synchronization to separate the signals at a receiver. We use this technique to transmit a sum of chaotic signals. We add the signals so that the DC components are canceled.

General layout

Figure 1 is a block diagram of our technique applied to a pair of 3-dimensional chaotic systems. Drive systems A and B do not have to be identical, although in this paper we will use identical systems for simplicity. A and B are both symmetric nonlinear systems (they do not have to be chaotic) with 2 attractors each. We form a linear combination of signals from A and B. We choose the linear combination so that the DC level of the transmitted signal u is 0. If we have identical systems in opposite attractors, this requires that $k_4 = k_1$, $k_5 = k_2$, and $k_6 = k_3$. For non identical systems, the k's must be chosen appropriately so that the time average of u is 0. The idea of making a linear combination of drive variables comes from control theory [16] and the work of Peng et al. [17] who used this technique to synchronize hyperchaotic systems. We have shown [18] that such a technique can make the response system very stable and insensitive to parameter mismatch.

To change symbols, we can either flip the attractors in A and B or we can invert the transmitted signal u, which is equivalent. In some cases, changing the attractors might have advantages, but here we choose the simpler approach of multiplying the transmitted signal u by $s = \pm 1$ to produce $u_s = su$. The signal s is our binary information signal.

The response systems are A' and B'. We make an identical linear combination of variables from A' and B' to make u' , and generate a difference signal $v = u_s - u'$. The difference signal is multiplied by one of the constants b_i (i = 1,6) and fed back into the response systems. The *k's* and *b's* are chosen by using a numerical minimization routine [19] to minimize the largest Lyapunov exponent for the response system.

Complications of multiplexing

Tsimring and Suschick [12] found that when they multiplexed chaotic signals from several maps and tried to separate the signals using chaotic synchronization, the response systems had large local instabilities that caused bursting away from synchronization when noise was added to the transmitted signal. We see the same effect in our systems. We show below that multiplexing by using linear combinations of chaotic signals with diffusive coupling always results in an unstable response system when the response system consists of two identical chaotic systems. We will show that under some conditions it might be possible to use non identical response systems that will be stable, although we do not know of any such systems.

The general setup is as follows,

$$\begin{aligned}
\frac{d\mathbf{x}^{(d1)}}{dt} &= \mathbf{F}^{(d1)}(\mathbf{x}^{(d1)}) \\
\frac{d\mathbf{x}^{(d2)}}{dt} &= \mathbf{F}^{(d2)}(\mathbf{x}^{(d2)}) \\
u &= \mathbf{K}^{(1)^T}\mathbf{x}^{(d1)} + \mathbf{K}^{(2)^T}\mathbf{x}^{(d2)} \\
\frac{d\mathbf{x}^{(r1)}}{dt} &= \mathbf{F}^{(r1)}(\mathbf{x}^{(r1)}) + \mathbf{B}^{(1)}\mathbf{K}^{(1)^T}(\mathbf{x}^{(d1)} - \mathbf{x}^{(r1)}) + \mathbf{B}^{(1)}\mathbf{K}^{(2)^T}(\mathbf{x}^{(d2)} - \mathbf{x}^{(r2)}) \\
\frac{d\mathbf{x}^{(r2)}}{dt} &= \mathbf{F}^{(r2)}(\mathbf{x}^{(r2)}) + \mathbf{B}^{(2)}\mathbf{K}^{(2)^T}(\mathbf{x}^{(d2)} - \mathbf{x}^{(r2)}) + \mathbf{B}^{(2)}\mathbf{K}^{(1)^T}(\mathbf{x}^{(d1)} - \mathbf{x}^{(r1)})
\end{aligned} \quad , \qquad (1)$$

where d and r label drive and response, respectively, u is the signal sent to the response systems, and $\mathbf{B}^{(i)}$ and $\mathbf{K}^{(i)}$ are the coupling vectors for system pairs i=1 and 2. The goal is to synchronize $d1$ with $r1$ and $d2$ with $r2$ in a stable fashion. The coupling is typical of what is used in control theory [16] and it was introduced by Peng *et al.* [17] to first prove that synchronizing hyperchaotic systems with scalar signals is possible. We have used such coupling to synchronize hyperchaotic maps [20, 21] and to mitigate against parameter mismatch [18]. It is the most general form of linear coupling using a scalar signal.

To establish our goal we examine the variational equation of the response system, shown below in block form.

$$\begin{pmatrix} \dfrac{d\xi^{(r1)}}{dt} \\ \dfrac{d\xi^{(r2)}}{dt} \end{pmatrix} = \left[\begin{pmatrix} \mathbf{J}^{(r1)} & 0 \\ 0 & \mathbf{J}^{(r2)} \end{pmatrix} + \begin{pmatrix} \mathbf{B}^{(1)} \\ \mathbf{B}^{(2)} \end{pmatrix} \begin{pmatrix} \mathbf{K}^{(1)T} & \mathbf{K}^{(2)T} \end{pmatrix} \right] \cdot \begin{pmatrix} \xi^{(r1)} \\ \xi^{(r2)} \end{pmatrix} \tag{2}$$

Note that in eq. (2) the product of the **B** and **K** vectors is an outer product which results in a coupling matrix **C** in block form,

$$\mathbf{C} = \begin{pmatrix} \mathbf{B}^{(1)}\mathbf{K}^{(1)T} & \mathbf{B}^{(1)}\mathbf{K}^{(2)T} \\ \mathbf{B}^{(2)}\mathbf{K}^{(2)T} & \mathbf{B}^{(2)}\mathbf{K}^{(1)T} \end{pmatrix}, \tag{3}$$

but we prefer to leave the coupling expressions in the outer product form since that will make the analysis clearer. What we want to do first is to block diagonalize eq. (2) to isolate the transverse blocks. If we can then show each block is stable we are done.

If the systems are identical the Jacobians $\mathbf{J}^{(r1)}$ and $\mathbf{J}^{(r2)}$ are also identical which makes the first term in eq. (2) a multiple of the 2 x 2 identity matrix. Hence, in this case we only need focus on the second term, the **BK**-outer-product matrix, to determine an eigenvalue block structure of the variational equation. The outer product structure immediately yields two eigendirections and their associated eigenvalues. Let $\chi_1=(\mathbf{B}^{(1)}, \mathbf{B}^{(2)})^T$, then we easily see that this is an eigenvector of the **BK**-outer-product matrix with the eigenvalue $\mathbf{K}^{(1)T}\mathbf{B}^{(1)}+\mathbf{K}^{(2)T}\mathbf{B}^{(2)}$. With the right choice of **B**'s and **K**'s we usually can make the Lyapunov exponents of this block matrix negative.

The other block is associated with the vector $\chi_2=(-\mathbf{K}^{(2)}, \mathbf{K}^{(1)})^T$ and its eigenvalue is 0. We can see that the 0 eigenvalue results from an eigenvector χ_2 that is orthogonal to the $(\mathbf{K}^{(1)}, \mathbf{K}^{2)})^T$ vector. When combined with the Jacobian matrix this leaves a variational block that has the same stability as an isolated, uncoupled system, like the drive. Since we are dealing with chaotic systems this means we cannot stabilize the synchronous state. The 0 eigenvalue is a result of the structure of the coupling matrix and so is generic for this form of coupling.

If we examine a multiplexing setup with many drives and responses the situation worsens for synchronization stability. In this case for N drives and N responses the coupling signal is given by $u = \sum_{i=1}^{N} \mathbf{K}^{(i)T}\mathbf{x}^{(di)}$ and the coupling matrix is the outer-product matrix,

$$\mathbf{C} = \begin{pmatrix} \mathbf{B}^{(1)} \\ \mathbf{B}^{(2)} \\ \vdots \\ \mathbf{B}^{(N)} \end{pmatrix} \begin{pmatrix} \mathbf{K}^{(1)T} & \mathbf{K}^{(2)T} & \cdots & \mathbf{K}^{(N)T} \end{pmatrix} \tag{4}$$

In analogy with our 2-drive system above the vector $\chi_1=(\mathbf{B}^{(1)}, \mathbf{B}^{(2)},...,\mathbf{B}^{(N)})^T$ is an eigenvector with eigenvalue $\sum_i \mathbf{K}^{(i)T}\mathbf{B}^{(i)}$. We also have N–1 vectors χ_i , i = 2,...,N–1 which are orthogonal to $(\mathbf{K}^{(1)}, \mathbf{K}^{(2)},...,\mathbf{K}^{(N)})^T$ and, hence, there are N–1 zero eigenvalues. This results in N–1 blocks whose stability is the same as the isolated system and, again, we cannot stabilize the synchronous state.

We note that there may be some relief from the situations above in using slightly different systems. Assume that the associated drives and responses (d1–r1 and d2 –r2) are identical, but $\mathbf{F}^{(d1)}$ and $\mathbf{F}^{(d2)}$ are different. This leads to different Jacobians. Let $\Delta\mathbf{J}$ be one-half the difference between the Jacobians, so that $\mathbf{J}^{(1)}=\mathbf{J}+\Delta\mathbf{J}$ and $\mathbf{J}^{(2)}=\mathbf{J}-\Delta\mathbf{J}$. This will perturb the λ_2 = 0 eigenvalue and if $\Delta\mathbf{J}$ is not too large we can estimate that perturbation using first-order perturbation theory [22, 23]. This give a new eigenvalue of $\lambda_2' = \lambda_2 + \mathbf{K}^{(2)T}\,\Delta\mathbf{J}\,\mathbf{K}^{(2)} - \mathbf{K}^{(1)T}\,\Delta\mathbf{J}\,\mathbf{K}^{(1)}$, where λ_2 an the eigenvalue of the isolated system. If we can adjust the vector fields so that $\Delta\mathbf{J}$ is positive (or negative) definite, then with the right choices of $\mathbf{K}^{(1)}$ and $\mathbf{K}^{(2)}$ we may be able to cause λ_2' to become negative leading to stability of the synchronous state, *providing* we can simultaneously maintain the stability of the block associated with eigenvector χ_1. We have not attempted to do this here, but instead we concentrate on systems that are in discernibly different states, depending on the drive systems, although there is no synchronization.

Non-synchronous response systems

Our response systems A' and B' actually do not synchronize to the drive systems A and B. In our case, this lack of synchronization is not a problem because we are not interested in synchronization itself but rather in determining which attractors the drive systems are in. We will see below that even without synchronization, we have more than enough information to determine the drive system attractors.

If our response system will never be stable, how is it possible to keep the motion of the response system bounded? One would guess that for certain parameter combinations, the unstable response system might have no attractors at all. While we know that the largest Lyapunov exponent for the response system can never be negative, there are some concepts used in the calculation of Lyapunov exponents that can be useful.

If we were only interested in the largest Lyapunov exponent for a multidimensional system, we would only need to know how a small perturbation vector changed in time. One common approach for estimating Lyapunov exponents is: 1) Find a Jacobian **J** for the response system; 2) form a randomly chosen unit vector $\xi(0)$ and multiply the vector by the Jacobian. The result is a derivative vector that describes how the unit vector will change in one time step ($d\xi/dt = \mathbf{J} \cdot \xi$); 3) using some integration algorithm, use the derivative vector to find a new vector $\xi(t)$ from the unit vector $\xi(0)$; 4) after some number of integration steps, compare the magnitude of the new vector to the original unit vector. The ratio $||\xi(t)||/||\xi(0)||$ (for a properly chosen integration time) will be $e^{\lambda t}$, where λ is the largest Lyapunov exponent for the response system.

If the eigenvectors for the response system are not orthogonal, then this calculation can yield misleading results. It is known [2, 24] that for a stable response system, the magnitude of the difference between drive and response can increase (burst) before shrinking. We find that for the type of response systems that we describe here, even though the response system is unstable, the magnitude of the difference between drive and response can shrink before growing.

We show a simple example, using the Jacobian of the system described in eq. (6). We use only the Jacobian for the variables x_1 and x_4 less than 3 and greater than -3, so that the Jacobian is constant. We set the k and b parameters as: $k_1 = 0.5352977$, $k_2 = 1.19088$, $k_3 = 0.2072838$, $k_4 = k_1$, $k_5 = k_2$, $k_6 = k_3$, $b_1 = 6.61433$, $b_2 = -0.463105$, $b_3 = 0$, $b_4 = 0.583074$, $b_5 = -1.11178$, $b_6 = 0$. The magnitude of the largest eigenvalue for the response Jacobian for these parameters is 1.84, indicating that the response system will not synchronize to the drive.

We begin with the unit vector (1/√6)(1,1,1,1,1,1) = v(0). We multiply by the Jacobian J to get J· v(0) = dv(0)/dt. We use the simplest possible integration algorithm to update v(0): v(1) = v(0) + τ (dv(0)/dt), where τ is the time step. We arbitrarily use a time step of τ = 0.1. We then iterate this process to produce new vectors v(n).

Figure 2 shows the value of v(n) for values of n from 0 to 99. We see that even for an unstable response, the value of v(n) may decrease as well as increase. Eventually, the drive and response systems do diverge. The results in Fig. 2, were very dependent on the time step used, so Lyapunov exponent calculations based on vector differences are not reliable for the type of system shown in Fig. 1.

The Lyapunov exponent calculation does have some use in setting up a communications system. By minimizing the length of the vector v(n) for some large n, we guarantee that the difference between drive and response systems will not grow too large in a short time. We are able to insure that our feedback signal does not destabilize the response system by too much, so the response still has the same two attractors as the drive.

Our communication system does not depend on systems A or B being chaotic. Periodic nonlinear systems may also have multiple attractors. We do require that A and B not be in the same state at the same time; otherwise, the transmitted signal u will be 0. Using chaotic systems for A or B prevents any accidental locking between A and B caused by spurious coupling. If A and B were periodic systems that were locked in phase or frequency, then the signal u might be zero or have some DC offset.

Symmetric Rossler system

Two Attractor Signaling

We first use a 3-dimensional system that is similar to the Rossler system. Our symmetric Rossler system has a symmetric piecewise nonlinearity. The system is described by:

$$i = 0,1$$

$$\frac{dx_{3i+1}}{dt} = -\alpha_i\left[0.05x_{3i+1} + 0.5x_{3i+2} + x_{3i+3}\right]$$

$$\frac{dx_{3i+2}}{dt} = -\alpha_i\left[-x_{3i+1} - \rho x_{3i+2}\right]$$

$$\frac{dx_{3i+3}}{dt} = -\alpha_i\left[x_{3i+3} - g_1\left(x_{3i+1}\right)\right]$$

$$g_1(x) = \begin{Bmatrix} 5(x+3) & x < -3 \\ 0 & -3 \le x \le 3 \\ 5(x-3) & x > 3 \end{Bmatrix} \qquad (5)$$

$$u = \sum_{j=1}^{6} k_j x_j$$

where ρ = 0.25, α_0 = 1.2 and α_1 = 1. The *k* parameters are given below. There are two chaotic systems (*i* = 0 and *i* = 1) corresponding to A and B in Fig. 1. Each drive system was in an opposite attractor.

We numerically integrated eqs. (5) with a 4-th order Runge-Kutta integration routine [19] with a time step of 0.2 s. Figure 3(a) shows the one of the attractors from the symmetric Rossler system, while Fig 3(b) shows the other. We will call these two attractors the + attractor and the - attractor. Figure 4 shows the transmitted signal *u*, showing that the time average of *u* is zero.

The response system is described by

$$
\begin{aligned}
& i = 0,1 \\
& u_s = su \qquad s = \pm 1 \\
& v = u_s - u' \\
& \frac{dx'_{3i+1}}{dt} = -\alpha_i\left[0.05x'_{3i+1} + 0.5x'_{3i+2} + x'_{3i+3} - b_{3i+1}v\right] \\
& \frac{dx'_{3i+2}}{dt} = -\alpha_i\left[-x'_{3i+1} - \rho x'_{3i+2} - b_{3i+2}v\right] \\
& \frac{dx'_{3i+3}}{dt} = -\alpha_i\left[x'_{3i+3} - g_1\left(x'_{3i+1}\right) - b_{3i+3}v\right]
\end{aligned}
\tag{6}
$$

$$
u' = \sum_{j=1}^{6} k_j x'_j
$$

where the parameters are the same as in eq. (5). The parameter $s = \pm 1$ is our binary information signal, as described above. The *k's* were $k_1 = -1.2824$, $k_2 = 1.91712$, $k_3 = 1.19166$, $k_4 = k_1$, $k_5 = k_2$, and $k_6 = k_3$ and the *b's* were $b_1 = 1.09793$, $b_2 = 0.65328$, $b_3 = 0$, $b_4 = 1.62025$, $b_5 = 1.12384$, $b_6 = 0$. We chose the *k's* and *b's* by minimizing the largest finite time Lyapunov exponent for the response system using 1000 time steps at 0.2 s per time step, although as we mention above, the result of our calculation is not a true Lyapunov exponent.

To decode the transmitted message, we simply track whether system A' is in the + attractor or the - attractor. To aid in the detection, we use a low pass filter. We can change the value of s at time $t = nT$, where T is one clock period. We assume that the clock in the receiver is already synchronized to the transmitter clock. Most performance calculations for binary modulation techniques are done assuming clock synchronization has been achieved [14]. Our detector is described by

$$
\begin{aligned}
& \frac{dw}{dt} = x'_3 - w \\
& n = 0,1,2,\ldots \\
& \textbf{if } t = nT \textbf{ then } w = 0
\end{aligned}
\tag{7}
$$

At time nT , just before resetting, a positive value of w indicates that $s = +1$, while a negative value of w indicates $s = -1$.

Figure 5 shows the detection process. Figure 5(a) shows the value of s with T = 8s. Figure 5(b) shows the value of x'_3 from eqs. (6), while Fig 5(c) shows the detector output w .

We characterized the performance of our communications system when subject to noise by calculating the probability of bit error P_b as a function of the ratio of bit energy E_b to noise power spectral density N_0 . We integrated eqs. (5-7) for 800,000 steps with a time step of 0.2 s. We set the value of s at +1 and reset the detector variable w to 0 every T = 40 s. We measured the value of w just before resetting. If the value of w was not greater than 0, a bit error was recorded.

We added Gaussian noise to the transmitted signal u . We changed the variance of the noise to change the power spectral density N_0 . We calculated the bit energy P_b by finding the average power in the transmitted signal and multiplying by the data period T. In Figure 6 we plot the bit error rate P_b as a function of E_b/N_0 . For comparison, we also plot P_b for a bipolar binary baseband signaling system, as calculated in [14]. A "bipolar binary baseband" signal consists of sending +1 or -1, as if we were transmitting only s.

Parameter Modulation Signaling

As an additional comparison in Fig. 6, we used the system of eqs. (5-7) to transmit information using parameter modulation. We switched the parameter ρ in the transmitter between 0.25 and 0.2, while keeping all parameters in the receiver fixed. When ρ was 0.2, the transmitter and receiver were not matched, and so did not synchronize. We used the error signal v in eq. (6) to detect the information signal. Our detector was again a low pass filter that used the square of v :

$$\begin{aligned} &\frac{dw}{dt} = v^2 - w \\ &n = 0,1,2,\ldots \\ &\textbf{if } \; t = nT \; \textbf{ then } w = 0 \end{aligned} \tag{8}$$

The bit period T was 200 s. We added Gaussian white noise to the transmitted signal as before and ran numerical simulations to find the bit error probability P_b as a function of E_b/N_0 . We plot these results in Fig. 6. From the figure, we can see that to achieve a given bit error probability, using two attractor signaling requires about 40 dB (a factor of 10,000) less energy per bit.

Alternate Circuit.

We have seen above that using two attractor signaling performs better than parameter modulation signaling. The symmetric piecewise linear Rossler system we used above is actually not the best system to use for two attractor signaling. The + and - attractors in the Rossler system that we used are not well separated. We might be able to get a communications system that was more robust to noise if we used a system with attractors that were farther apart. We describe such a system below, where the average of the measured signal x'_4 was 1.47, compared to an average value of 0.29 for the measured signal x'_3 in the piecewise linear Rossler system of eq. (6).

Our improved communications system uses a four dimensional chaotic system, described by

$$\begin{aligned}
\frac{dx_1}{dt} &= -\left[1.5x_1 - 2.4x_2 + x_3 - g_2(x_4)\right] \\
\frac{dx_2}{dt} &= -10\left[x_1 + 0.2x_2\right] \\
\frac{dx_3}{dt} &= -4.5\left[x_2 + 0.2x_3\right] \\
\frac{dx_4}{dt} &= -10\left[-2.5x_1 + 0.5x_4 + g_3(x_4)\right] \\
g_2(x) &= \left\{\begin{matrix} -0.4 & x \le -0.4 \\ 0 & -0.4 < x < 0.4 \\ 0.4 & x \ge 0.4 \end{matrix}\right\} \\
g_3(x) &= \left\{\begin{matrix} 0.5x+1 & x < -2 \\ -0.5x & -2 \le x \le 2 \\ 0.5x-1 & x > 2 \end{matrix}\right\}
\end{aligned} \quad . \quad (9)$$

This system is similar to the system used in [15], except that the g_3 function has been simplified.

Figure 7(a) is a plot of the + attractor for this system, while Fig. 7(b) is a plot of the - attractor. There is a better separation between the + and - attractors than there was for the + and - attractors in the symmetric Rossler system of eq. (5).

We use a pair of these 4-dimensional chaotic systems in the same manner as the example of Fig. 1. The receiver uses two copies of the 4-D system with coupling constants k_1 = -1.14018, k_2 = 1.17253, k_3 = -0.714629, k_4 = -0.45176, $k_5 = -k_1$, $k_6 = -k_2$, $k_7 = -k_3$, $k_8 = -k_4$, $b_1 = 0$, $b_2 = 1/k_1$, $b_3 = 0$, $b_4 = 1/k_1$, $b_5 = b_1$, $b_6 = b_2$, $b_7 = b_3$ and $b_8 = b_4$. Once again, the transmitted signal has no DC component.

We include the probability of bit error P_b as a function of bit energy/noise spectral density for the system of eq. (9) on the plot in Fig. 6. We can see that using widely separated attractors does improve the noise resistance of our two-attractor communications system. With some small improvements in our signaling scheme, we are approaching the efficiency of existing digital signalling techniques.

Conclusions

Tsimring and Suschick noted that there was a maximum number of maps that they could synchronizing using their multiplexing technique, and they could not see synchronization in coupled flow systems. Our results here explain why they could not see synchronization in flows.

We have shown that using attractors for symbols can improve the noise robustness of a chaotic communications system by several orders of magnitude. Our receiver does not have to be synchronized to the transmitter, it only has to tell us which attractor the transmitter is in. A two attractor chaotic communication system can approach the noise robustness of conventional communications systems.

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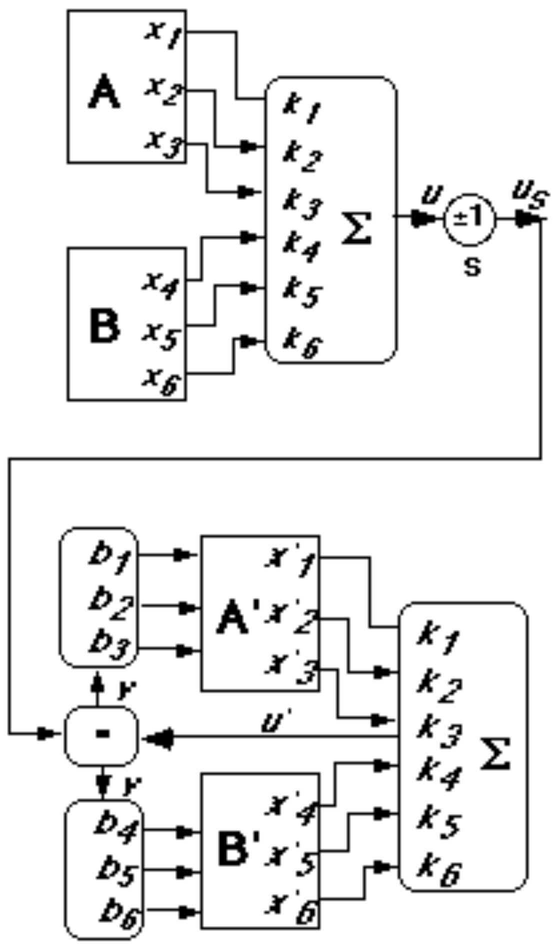


Fig. 1 Block diagram of a 2-attractor chaotic communications system. A and B are chaotic systems in the transmitter, while A' and B' are chaotic systems in the response. The information signal, *s*, is ± 1.

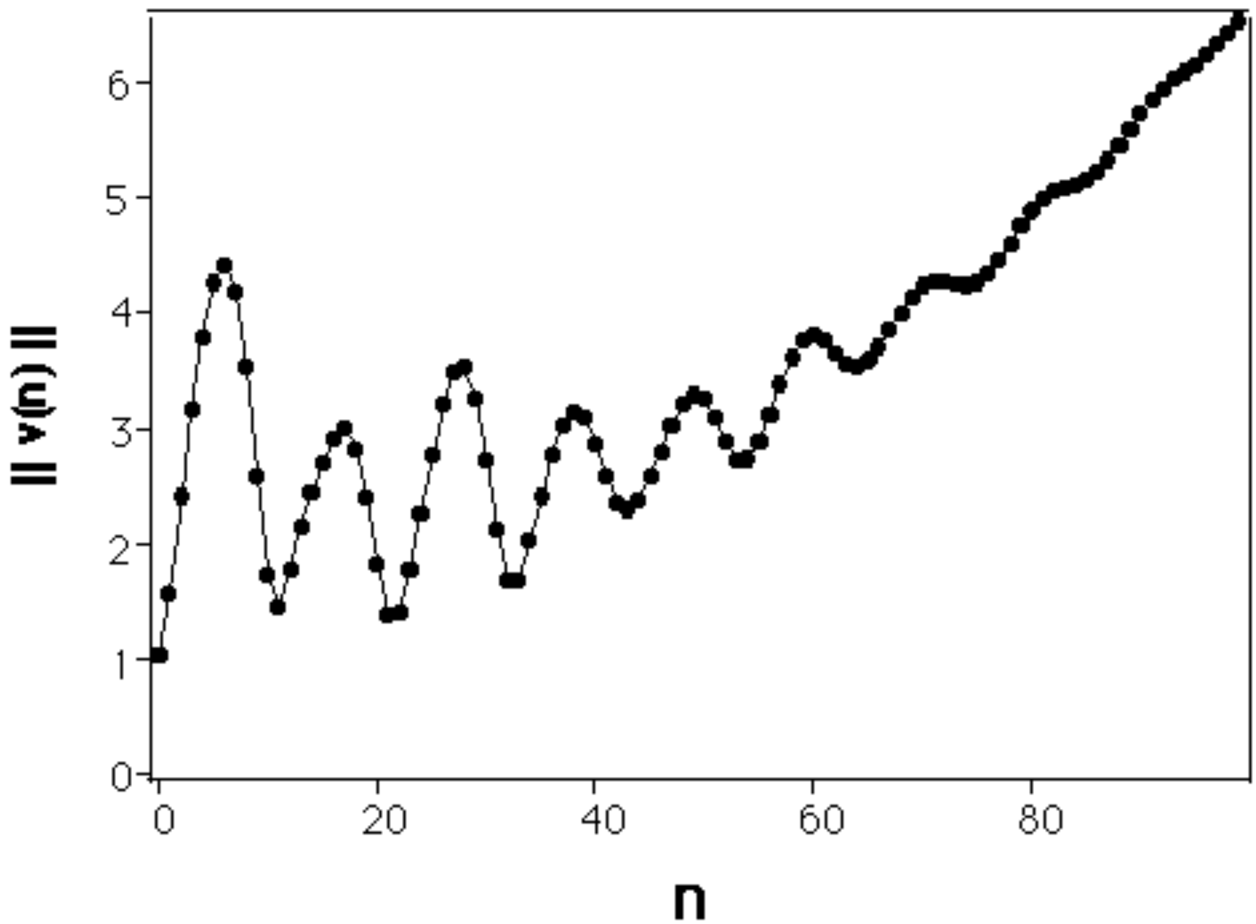
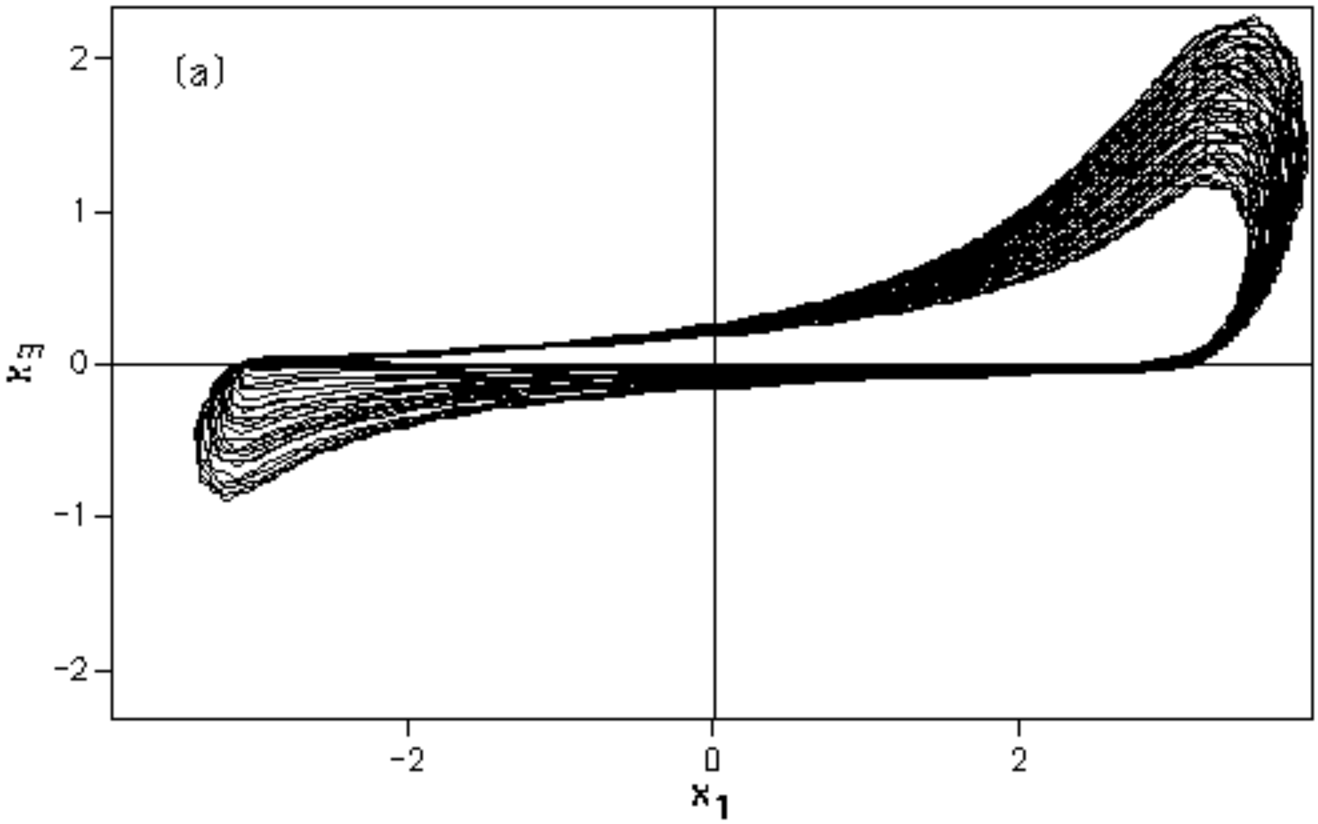


Fig. 2. Magnitude of distance vector ***v*** *(n)* between chaotic response and drive systems, as determined from the Jacobian of eqs. (6). Although the response system is unstable, for some intervals the response vector can shrink as well as grow, possible giving a false result in a Lyapunov exponent calculation.


(a)
2
1
0
-1
-2
x_3
-2
0
2
x_1

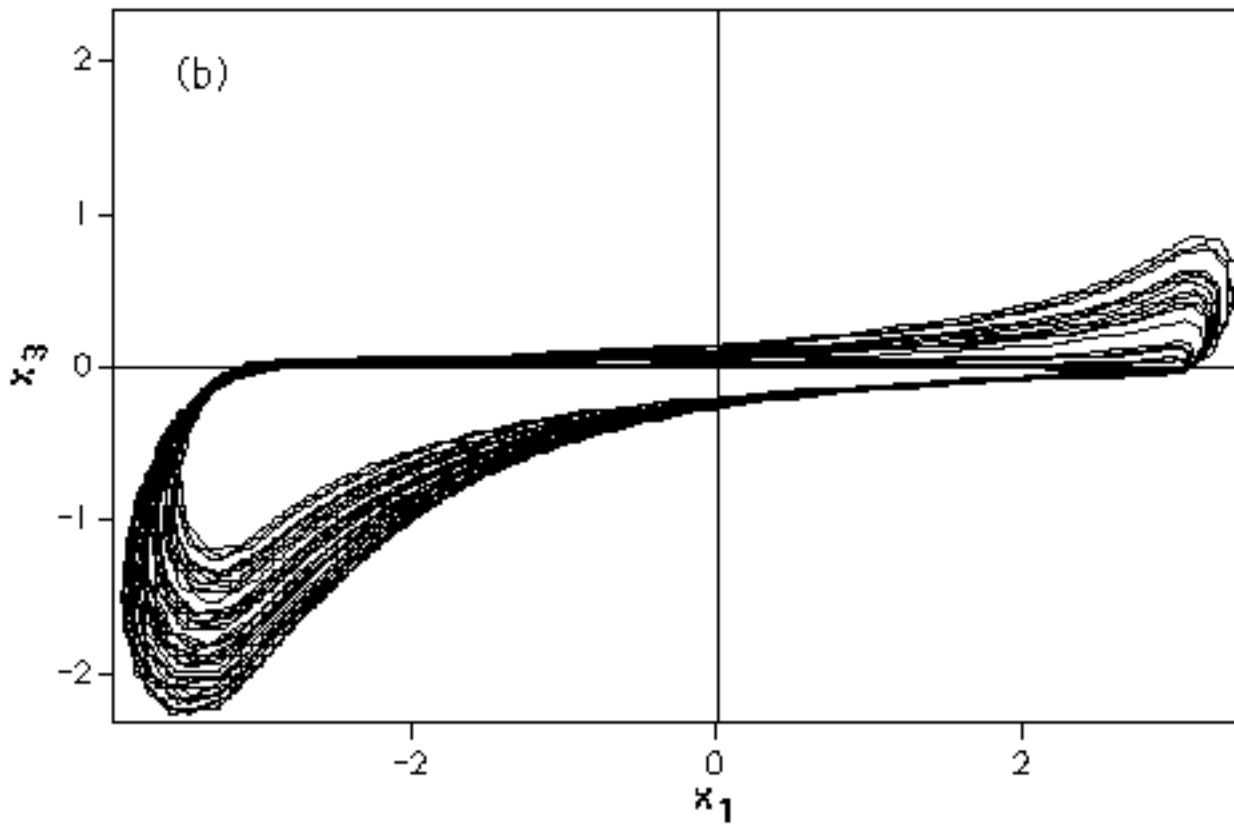


Fig. 3 (a) is the + attractor for the symmetric Rossler system of eq. (5). (b) is the - attractor for the symmetric Rossler system of eq. (5).

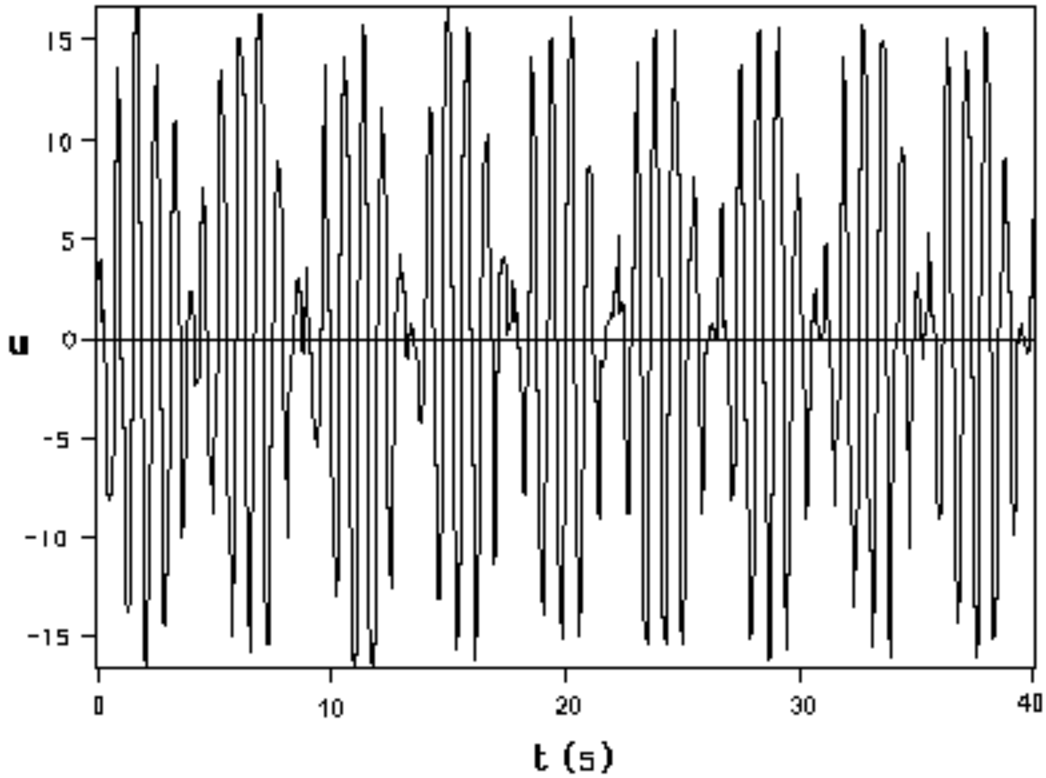


Fig.4 Transmitted signal u from the symmetric Rossler system, before being multiplied by the information signal s . The modulation of the signal is caused be the difference between the center frequencies of the two Rossler systems.

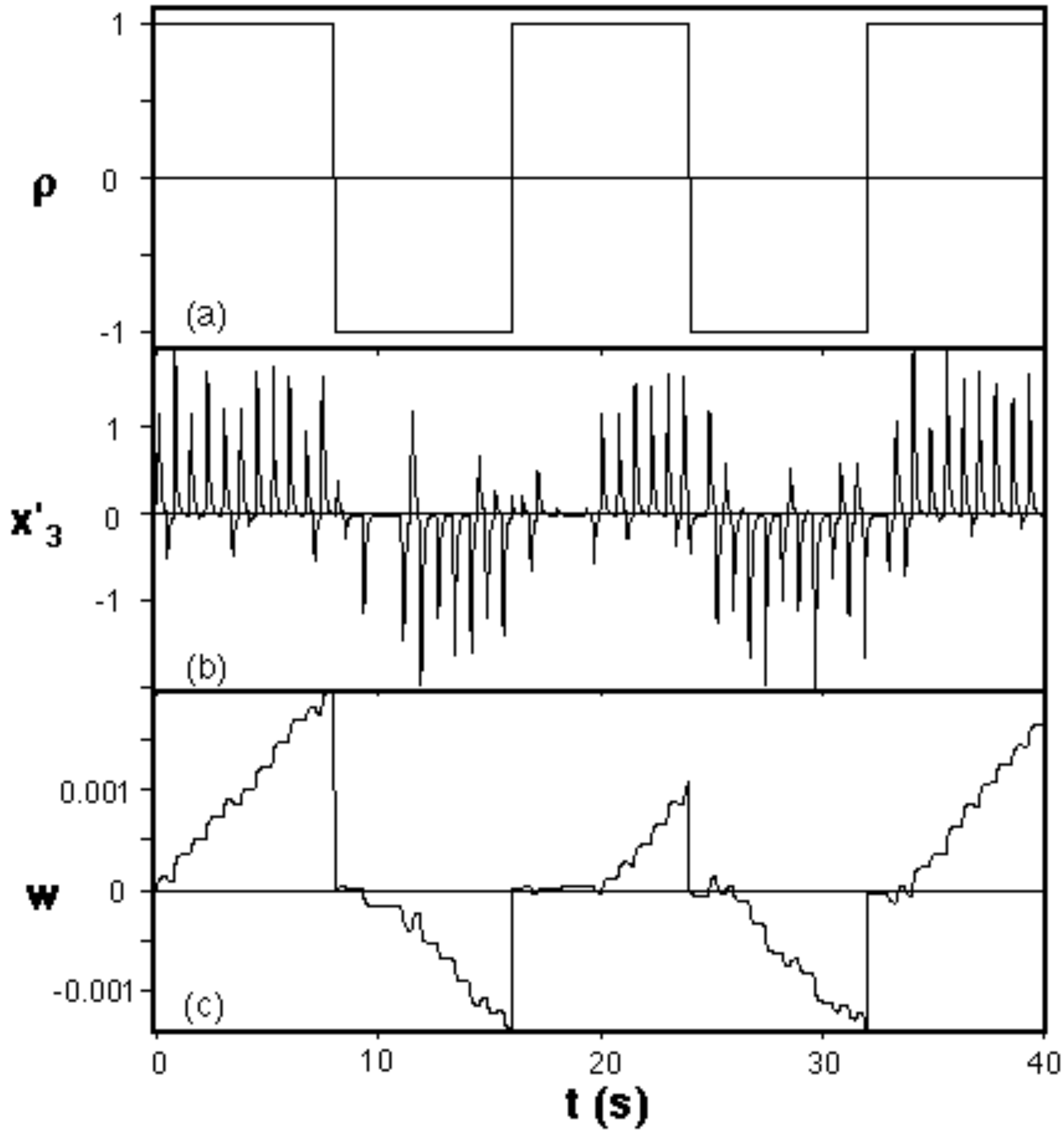


Fig. 5 (a) is the information signal *s* . (b) is the signal x'_3 from the response system. (c) is *w* , the output of the signal detector.

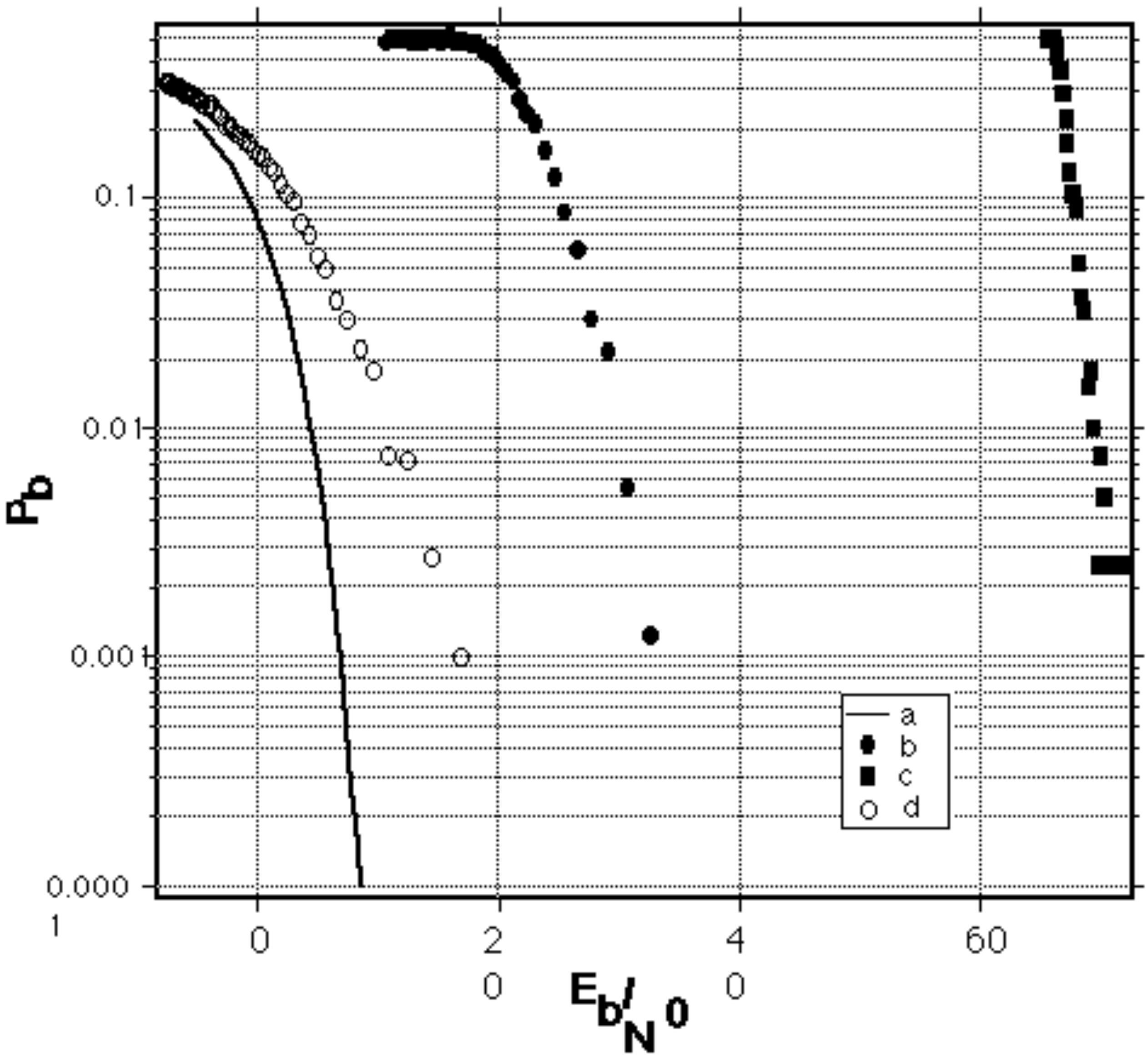
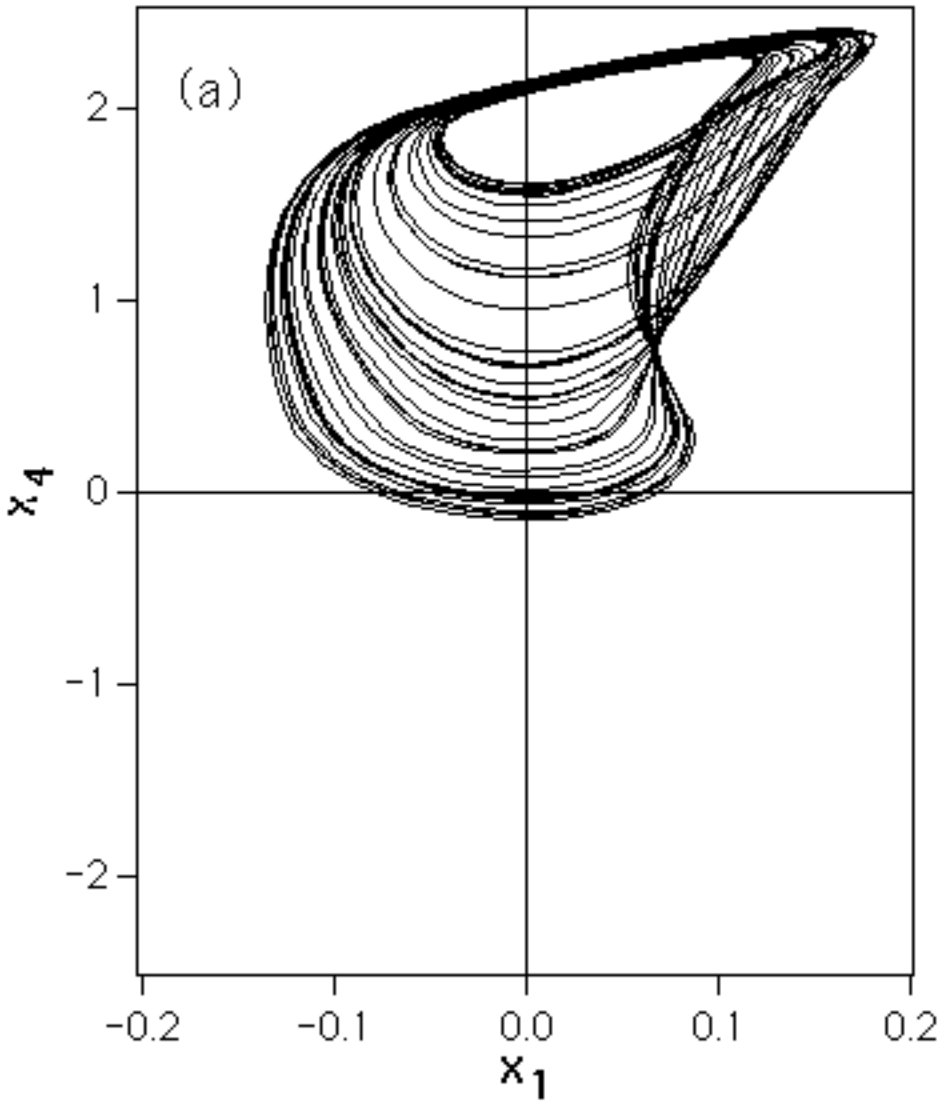


Fig. 6 Probability of bit error P_b as a function of energy per bit E_b divided by noise power spectral density N_0 for several different communications systems. (a), the solid line, is an analytic example for a bipolar baseband signal from [14], shown for comparison. (b), the solid circles, is the result for the symmetric Rossler system of eqs (5-8) using two attractor signaling. (c), the solid squares, is the result for the symmetric Rossler system of eqs. (1-3) using parameter modulation. (c), the open circles, is the result for the 4-D system of eq. (5) using two attractor switching.


(a)
2
1
0
-1
-2
x_4
-0.2
-0.1
0.0
0.1
0.2
x_1

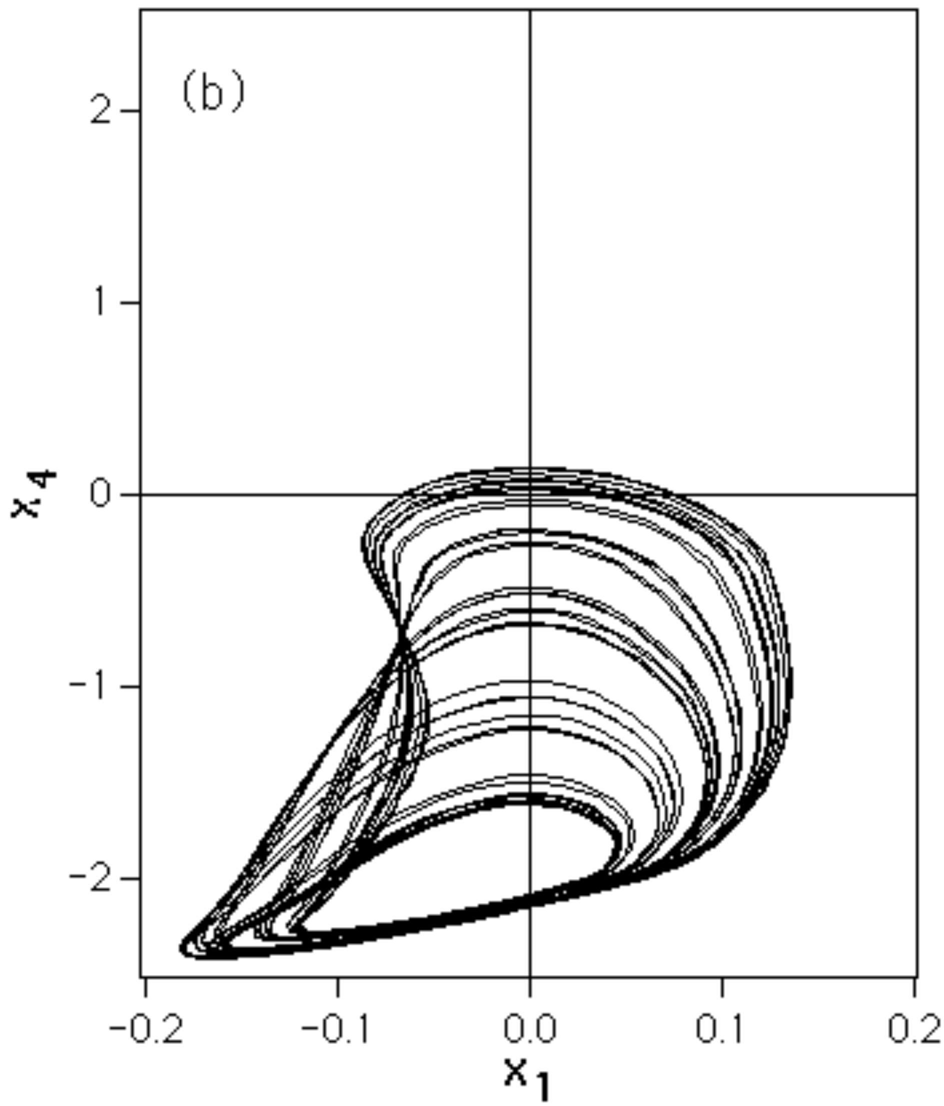


Fig. 7(a) The + attractor for the 4-D system of eq. (9). (b) The - attractor for the 4-D chaotic system of eq. (9).